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Internal Symmetry of Hadrons: Finsler Geometrical Origin

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The microlocal space of hadronic matter extension has recently been characterized as a Finsler space. This consideration of hadrons extended as composites of constituents can give rise to a dynamical theory of hadrons. The macrospaces, the space-time of common experience (the Minkowski flat space-time) and the Robertson–Walker background space-time of the universe, are found to appear as the "averaged" space-times of the Finsler space that describes the anisotropic nature of the microdomain of hadrons. From the assumed property of the fields of the constituents in the microspace it is possible to find the field (or wave) equations of the particles (or constituents) through the quantization of space-time at small distances (to an order of or less than a fundamental length). If the field (or wave) function is separable in the functions of the coordinates of the underlying manifold and the directional arguments of the Finsler space, then the former part of the field function is found to satisfy the Dirac equation in the Minkowski space-time or in the Robertson–Walker space-time according to the nature of the underlying manifold. In the course of finding a solution for the other part of the field function a relation between the mass of the particle and a parameter in the metric of the space-time has been obtained as a byproduct. This mass relation has cosmological implications and is relevant in the very early stage of the evolution of the universe. In fact, it has been shown elsewhere that the universe might have originated from a nonsingular origin with entropy and matter creations that can account for the observed photon-to-baryon ratio and total particle number of the present universe. The equations in the directional arguments for the constituents in the hadron configuration are found here and give rise to an additional quantum number in the form of an "internal" helicity that can generate the internal symmetry of hadron if one incorporates the arguments of Budini in generating the internal isospin algebra from the conformal reflection group. This consideration can also account for the meson–baryon mass differences.

1. INTRODUCTION

Hadrons have recently been considered as extended objects in a microlocal anisotropic space-time (De, 1986a,b, 1989, 1991). This anisotropic micro-

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domain of hadronic matter extension has been characterized as Finslerian in nature. The motivation behind such considerations is to accomplish a space-time formulation of the internal symmetry of hadrons. Apart from this, it is also necessary to provide a dynamical theory for the hadrons, and to this end, the hadronic particle states and fields have been constructed in the Finslerian inner space-time (the microdomain of hadronic matter extension). Consequently, it has been shown that in the field theory of hadrons the perturbation technique is applicable even in the strong interaction. Thus, the anisotropic character of this inner space-time is found to have a direct bearing in providing a consistent dynamics of the strong-interacting subatomic particles.

The Minkowski flat space-time, the macrospace of common experience, has also been recovered from the microdomain which is manifested in the length scale of the order of or less than a fundamental length (De, 1989). Specifically, the metric of Minkowski space-time can be obtained by a prescribed "averaging" on the metric of the Finsler space. Also, the field (or wave) equation in this anisotropic Finsler space has been derived from the assumed property of the field (or wave) functions and by space-time quantization at very small distances. It is shown there that if the field (or wave) function is separable in the functions of the coordinates of the underlying manifold and the directional arguments of the Finsler space, then the former function satisfies the usual Dirac equation in the Minkowski space-time.

The Dirac equation in the curved space-time, particularly for the Robertson–Walker (RW) background space-time of the universe, has also been derived (De, 1991). Moreover, it is possible to resurrect the RW metric that describes the large-scale structure of the universe from the more general metric of the microdomain. The equation satisfied by the other part of the field (or wave) function, which depends on the directional variables, has been solved. An important relation between the mass of the elementary particle and a parameter characterizing the metric tensor has also been obtained as a byproduct. In terms of the RW metric, that is, in the context of the evolution of the universe, the mass of the particle is found to have two distinct parts, one independent of time and the other "epoch-dependent." Of course, the latter part may be zero for some species of particles and for the constituent particles in the hadron configuration. This mass relation has cosmological consequences and in fact it was shown (De, 1993a) that the universe can have a nonsingular origin with matter and entropy productions in its very early stage of evolution. The matter and entropy productions were considered in the framework of thermodynamically open universe originally proposed by Prigogine (1989) with the incorporation of this mass relation. This phenomenological approach was also supplemented by a quantum mechanical consideration for the creation of matter in the Planck-era time. The totality of the produced particles and the photon-to-baryon number ratio as calculated in these approaches were seen to be in good agreement with the observational data. In addition to the consideration of the nonsingular origin of the universe from anisotropic perturbation of the Minkowski flat space-time, it is also argued (De, 1995) that the cosmological constant problem can be resolved if one adopts the changing gravity approach (Weinberg, 1989).

Besides discussing further the nature of Finsler space of hadronic matter extension, the deductions of the field (or wave) equation in that space, and the Dirac equation in RW space-time, the present consideration will focus on the generation of an extra quantum number, the "internal" helicity of hadron constituents. This, in turn, can give rise to the internal symmetry of hadrons if one takes into account the arguments of Budini (1979) and also subsequent discussions by Bandyopadhyay (1989).

We begin in Section 2 with a recapitulation of previous work on Finsler space with the additional feature regarding this special space that describes hadronic matter extension. In Section 3 the general field (or wave) equation in Finsler space will be discussed. In Section 4 the separation of the wave (or field) function is made and consequently the Dirac equation in RW space-time together with the mass relation are established. In Section 5 the internal helicity for the constituents of hadrons is obtained. In Section 6, following Budini's approach, SU_2 algebra (the internal isospin algebra) is generated and subsequently it is shown that the internal symmetry of hadrons can be obtained by incorporating this additional quantum number (internal helicity). This is followed by concluding remarks, particularly on meson–baryon mass differences.

2. FINSLER SPACE OF HADRONIC MATTER EXTENSION

Riemann (1854) suggested that the positive fourth root of a fourth-order differential form might serve as a metric function. The common property with the Riemannian quadratic form is that they are both positive, homogeneous of the first degree in the differentials, and also convex in the latter. Thus, the notion of distance between two neighboring points x^i and $x^i + dx^i$ can be generalized as given by a fundamental function $ds = F(x^i, dx^i)$ satisfying these three conditions. This program was in fact carried out by Finsler (1918) and subsequently developed by such mathematicians as Cartan (1934), Berwald (1941), Rund (1959), Matsumoto (1986), and Asanov (1985). More recently the theory of Finsler spaces has been applied in various branches of physics and biology (Antonelli *et al.*, 1993).

The fundamental function of the Finsler space, $F(x^i, dx^i)$, is positively homogeneous of degree one in dx^i, i.e.,

$$F(x^i, \lambda dx^i) = \lambda F(x^i, dx^i), \qquad \lambda > 0 \tag{2.1}$$

The second set of arguments is referred to as the directional arguments or variables.

From this fundamental function a metric tensor can be defined as (Rund, 1959; Asanov, 1985)

$$g_{ij}(x^k, \dot{x}^k) = \frac{1}{2} \frac{\partial^2 F^2(x^k, \dot{x}^k)}{\partial \dot{x}^i \, \partial \dot{x}^j} \tag{2.2}$$

Here the directional dependence is taken so as to be on the components $\dot{x}^i = dx^i/dt$ of a tangent vector to the manifold, namely, the one given by the curve $x^i(t)$. In general, it is defined for Finsler spaces that the fundamental functions $F(x, y)$ depend on the directional arguments or variables y^i which are tangent to the point x^i. It should be noted that spaces are possible for which **x** and **y** are independent (Beil, 1989, 1992). In fact, Asanov *et al.* (1988) and Asanov and Kiselev (1988) have considered a theory of gauge transformations in the context of Finsler space in which Finsler tangent vectors have been treated as independent variables attached to points in space-time.

Blokhintsev (1978) pointed out that the distance element of ordinary Minkowski flat space-time or Lorentz space can be regarded as a particular homogeneous Finsler space in the sense that one can write $ds = N_\mu \, dx^\mu$, where the vector N_μ is a zeroth-order form in $d\mathbf{x}$. This form is different for spacelike, timelike directions, and the light cone. In fact, there are three possible values $\mathbf{N}^2 = \mp 1$ and 0. Thus, ds depends on the direction $d\mathbf{x}$ in the sense that the spacelike, timelike, and lightlike directions are distinguished. This concept was exploited for characterization of the microlocal domain of hadronic matter extension (De, 1989, 1991). The metric tensor $g_{\mu\nu}(d\mathbf{x})$ depends (apart from the dependence on the **x** variable) upon the direction $d\mathbf{x}$ and the line element of this Finsler space is given by

$$ds^2 = F^2(\mathbf{x}, d\mathbf{x}) = g_{\mu\nu}(d\mathbf{x}) \, dx^\mu \, dx^\nu \tag{2.3}$$

A particular Finsler space is given by

$$g_{\mu\nu}(d\mathbf{x}) = \mathrm{diag}(1, -1, -1, -1) \qquad \text{for timelike directions}$$

$$= \mathrm{diag}(-1, +1, +1, +1) \qquad \text{for spacelike directions}$$

In fact, the Finsler spaces of this type are given by

$$F^2(\mathbf{x}, \boldsymbol{\nu}) = \eta_{ij} g(\mathbf{x}) f(\boldsymbol{\nu}) \nu^i \nu^j \tag{2.4}$$

with $g(\mathbf{x}) = \exp(-b_k x^k)$, $(b_k x^k)^n$, or $(1 + b_k x^k)^n$ or similar physically relevant functions of the coordinates of the underlying manifold; and

$$\eta_{ij} = \mathrm{diag}(1, -1, -1, -1)$$

$$f(\boldsymbol{\nu}) = \epsilon(\boldsymbol{\nu}^2) \hat{f}(\boldsymbol{\nu})$$

$$\boldsymbol{\nu}^2 = \nu_0^2 - \vec{\nu}^2$$
$$\epsilon(\boldsymbol{\nu}^2) = 1 \quad \text{for} \quad \boldsymbol{\nu}^2 \geq 0$$
$$= -1 \quad \text{for} \quad \boldsymbol{\nu}^2 < 0$$

In previous work (De, 1989, 1991) the anisotropy of hadronic matter extension, which is manifested in the order of a length scale given by a fundamental length l, was characterized as Finslerian with $\hat{f}(\boldsymbol{\nu}) = 1$. For this Finsler space the fundamental function is given by

$$F^2(\mathbf{x}, \boldsymbol{\nu}) = \hat{g}_{ij}(\mathbf{x}, \boldsymbol{\nu})\nu^i\nu^j \tag{2.5}$$
$$\hat{g}_{ij}(\mathbf{x}, \boldsymbol{\nu}) = \eta_{ij}g(\mathbf{x})\epsilon(\boldsymbol{\nu}^2)$$

The metric tensor of this space, $g_{ij}(\mathbf{x}, \boldsymbol{\nu})$, is

$$g_{ij}(\mathbf{x}, \boldsymbol{\nu}) = \hat{g}_{ij}(\mathbf{x}, \boldsymbol{\nu}) + 4g(\mathbf{x})\delta(\boldsymbol{\nu}^2)\eta_{\alpha i}\eta_{\beta j}\nu^\alpha\nu^\beta \tag{2.6}$$

where $\delta(\boldsymbol{\nu}^2)$ is the Dirac δ-function.

The metric tensor of the Minkowski flat space-time and also that of the conformal space-time to the Minkowski space which corresponds to the RW space-time have been shown to appear through a prescribed averaging procedure on the metric tensor of the Finsler space. In De (1989) this procedure as the integration over the tangent space ($\boldsymbol{\nu}$ space) was performed by using weight functions or probability density functions (pdf) of some specific types that are complex functions. Here we consider the following real pdf:

$$\varnothing(\boldsymbol{\nu}) = \frac{1}{8\pi^2}\,\boldsymbol{\nu}^2\epsilon(\boldsymbol{\nu}^2)e^{-\delta_{ij}\nu^i\nu^j/2} \tag{2.7}$$

The metric tensor for the macrospace can be found on integration over the tangent space with this weight function. It is given by

$$g_{ij}(\mathbf{x}) = \int g_{ij}(\mathbf{x}, \boldsymbol{\nu})\varnothing(\boldsymbol{\nu})\, d^4\nu$$
$$= \int \hat{g}_{ij}(\mathbf{x}, \boldsymbol{\nu})\varnothing(\boldsymbol{\nu})\, d^4\nu$$
$$= g(\mathbf{x})\eta_{ij} \tag{2.8}$$

If, for example, $g(\mathbf{x}) = \exp[-b_k(l)x^k]$, where the parameters $b_k(l)$ are functions of the fundamental length l such that $b_k(l) \to 0$ as $l \to 0$, then $g(\mathbf{x}) \to 1$ as $l \to 0$. Thus, $g_{ij}(\mathbf{x})$ becomes the metric tensor η_{ij} of the Minkowski flat space-time.

The RW background metric of the universe can also be deduced from the metric tensor given by (2.8) of the space-time conformal to the Minkowski space-time through a pure-time transformation if one takes

$$g(\mathbf{x}) = (1 + b_0 x^0)^n \quad \text{or} \quad (b_0 x^0)^n \qquad \text{with } x_0 = ct \tag{2.9}$$

Note that we have taken η_{ij} to be of signature $+2$. On the other hand, if η_{ij} has signature -2 in equation (2.5), then we have to choose the pdf as

$$\varnothing(\boldsymbol{\nu}) = \frac{1}{24\pi^2} \boldsymbol{\nu}^2 (\nu^0)^4 \epsilon(\boldsymbol{\nu}^2) e^{-\delta_{ij}\nu^i\nu^j/2} \tag{2.10}$$

to arrive at the same result. This result shows that the underlying manifold of the Finsler space (the $\mathbf{x}$ space) represents macrospaces such as the RW space-time.

Now, $g^{ij}(\mathbf{x}, \boldsymbol{\nu})$ can be seen to be

$$g^{ij}(\mathbf{x}, \boldsymbol{\nu}) = \hat{g}^{ij}(\mathbf{x}, \boldsymbol{\nu}) - \frac{4\delta(\nu^2)\nu^i\nu^j}{g(\mathbf{x})}$$

$$\hat{g}^{ij}(\mathbf{x}, \boldsymbol{\nu}) = \frac{\eta^{ij}\epsilon(\boldsymbol{\nu}^2)}{g(\mathbf{x})} \tag{2.11}$$

Also, the following important relations hold for the Finsler space which we are considering:

$$P^i_{jk}\nu^j\nu^k = \gamma^i_{jk}\nu^j\nu^k = \hat{\gamma}^i_{jk}\nu^j\nu^k \tag{2.12}$$

where $P^i_{jk}(\mathbf{x}, \boldsymbol{\nu})$ and $\gamma^i_{jk}(\mathbf{x}, \boldsymbol{\nu})$ are the connection coefficients and Christoffel symbols of second kind, respectively. $\hat{\gamma}^i_{jk}$ are the Christoffel symbols of second kind as calculated from $\hat{g}_{ij}$ and $\hat{g}^{ij}$ (these are not actually the metric tensors of the Finsler space, but are only related to them). Therefore, it is evident from (2.12) that the equation of an autoparallel curve or a geodesic in this Finsler space will be the same if one calculates it from $\hat{\gamma}^i_{jk}$, that is, from $\hat{g}_{ij}$ and $\hat{g}^{ij}$. It should be noted that this is, in general, not true for other types of Finsler spaces.

3. FIELD (WAVE) EQUATION IN FINSLER SPACE

As mentioned above, we are considering hadrons as extended objects in a microdomain or inner space-time which is a four-dimensional manifold. This manifold is anisotropic and is characterized as a Finsler space which is manifested in the order of or less than the length scale of a fundamental length l. Also, the hadronic matter extension is manifested as a composite particle of constituents (maybe leptons or quarks). It was indeed suggested (Bandyopadhyay, 1984; De, 1986b) that they might be leptons like μ^+, μ^-, ν_μ.

Now, we regard the field or wave functions of the constituents or particles as functions of the line support element $(\mathbf{x}, \boldsymbol{\nu})$ of the Finsler space. Thus, $\Psi(\mathbf{x}, \boldsymbol{\nu})$ is the wave (field) function in the Finsler space with metric tensor $g_{ij}(\mathbf{x}, \boldsymbol{\nu})$ and connection coefficients $P^{\mu}_{hj}(\mathbf{x}, \boldsymbol{\nu})$. The wave (field) equation for $\Psi(\mathbf{x}, \boldsymbol{\nu})$ can be obtained if one admits the following two conjectures:

1. *An equivalent property* to be satisfied by $\Psi(\mathbf{x}, \boldsymbol{\nu})$ along the neighboring points in the microdomain on the autoparallel curve, which is the curve whose tangent vectors result from each other by successive infinitesimal parallel displacements of the type

$$dv^i = -P^i_{hj}(\mathbf{x}, \boldsymbol{\nu})v^h\, dx^j \tag{3.1}$$

This property can be stated as

$$\delta\Psi = \{\Psi(\mathbf{x} + d\mathbf{x}, \boldsymbol{\nu} + d\boldsymbol{\nu}) - \Psi(\mathbf{x}, \boldsymbol{\nu})\} \propto \psi(\mathbf{x}, \boldsymbol{\nu})$$

or

$$\Psi(\mathbf{x} + d\mathbf{x}, \boldsymbol{\nu} + d\boldsymbol{\nu}) - \Psi(\mathbf{x}, \boldsymbol{\nu}) = \epsilon mc\Psi(\mathbf{x}, \boldsymbol{\nu}) \tag{3.2}$$

Here the mass term m appears as the constant of proportionality and may be regarded as the "inherent" mass of the particle. Also, ϵ is a real parameter such that $0 < \epsilon \leq l$.

2. *Quantization of the differentials dx^{μ} and hence also of dv^{μ}.* This is achieved with the change of coordinate differentials dx^{μ} by the finite operators $\Delta\hat{x}^{\mu} = i\epsilon\hbar\gamma^{\mu}$, where γ^{μ} ($\mu = 0, 1, 2, 3$) are Dirac matrices which may be either flat-space or curved-space matrices according to the nature of the underlying manifold of the Finsler space. This means that the underlying manifold $\hat{R}_4$ is quantized at small distances. A similar quantization of the space-time manifold of small distances has been considered by Namsrai (1985).

By the first conjecture it follows that

$$(dx^{\mu}\, \partial_{\mu} + dv^l\, \partial'_l)\Psi(\mathbf{x}, \boldsymbol{\nu}) = \epsilon mc\Psi(\mathbf{x}, \boldsymbol{\nu}) \tag{3.3}$$

where

$$\partial_{\mu} \equiv \partial/\partial x^{\mu} \qquad \text{and} \qquad \partial'_l \equiv \partial/\partial v^l$$

Now, the second conjecture can be considered in two steps:

1. The differentials dx^{μ} are quantized to

$$d\hat{x}^{\mu} = i\epsilon\hbar\gamma^{\mu}(\mathbf{x}) \tag{3.4}$$

It should be noted here that we are considering the underlying manifold to be curved space-time. This procedure makes $\Psi(\mathbf{x}, \boldsymbol{\nu})$ a spinor, i.e.,

$$\psi(\mathbf{x}, \boldsymbol{\nu}) \to \begin{pmatrix} \Psi_1(\mathbf{x}, \boldsymbol{\nu}) \\ \Psi_2(\mathbf{x}, \boldsymbol{\nu}) \\ \Psi_3(\mathbf{x}, \boldsymbol{\nu}) \\ \Psi_4(\mathbf{x}, \boldsymbol{\nu}) \end{pmatrix}$$

Consequently, we have the following equation for the spinor $\Psi(\mathbf{x}, \boldsymbol{\nu})$:

$$i\epsilon\hbar\gamma^{\mu}_{\alpha\beta}\partial_{\mu}\Psi_{\beta}(\mathbf{x}, \boldsymbol{\nu}) + d\nu^{l}\,\partial'_{l}\Psi_{\alpha}(\mathbf{x}, \boldsymbol{\nu}) = \epsilon mc\Psi_{\alpha}(\mathbf{x}, \boldsymbol{\nu}) \tag{3.5}$$

2. The differentials $d\nu^{l}$ are quantized by first noting that since the neighboring points $(\mathbf{x}, \boldsymbol{\nu})$ and $(\mathbf{x} + d\mathbf{x}, \boldsymbol{\nu} + d\boldsymbol{\nu})$ lie on the autoparallel curve of the Finsler space, the relation (3.1) between the differentials $d\nu^{l}$ and dx^{μ} holds. Therefore, the quantized differentials $d\hat{\nu}^{l}$ are

$$d\hat{\nu}^{l} = -i\epsilon\hbar P^{l}_{h\mu}(\mathbf{x}, \boldsymbol{\nu})\nu^{h}\gamma^{\mu}(\mathbf{x}) \tag{3.6}$$

and consequently the wave (field) function becomes a 'bispinor' (this nomenclature is only formal and for convenience) $\Psi_{\alpha\beta}(\mathbf{x}, \boldsymbol{\nu})$. Then the resulting equation for the wave (field) function $\Psi_{\alpha\beta}(\mathbf{x}, \boldsymbol{\nu})$ for the particle (or constituent) in the Finsler space is given by

$$\begin{aligned} &i\hbar\gamma^{\mu}_{\alpha\beta'}(\mathbf{x})\,\partial_{\mu}\Psi_{\beta'\beta}(\mathbf{x}, \boldsymbol{\nu}) - i\hbar\gamma^{\mu}_{\beta\beta'}(\mathbf{x})P^{l}_{h\mu}(\mathbf{x}, \boldsymbol{\nu})\nu^{h}\,\partial'_{l}\Psi_{\alpha\beta'}(\mathbf{x}, \boldsymbol{\nu}) \\ &\quad = mc\Psi_{\alpha\beta}(\mathbf{x}, \boldsymbol{\nu}) \end{aligned} \tag{3.7}$$

or, in compact form,

$$i\hbar\gamma^{\mu}(\mathbf{x})(\partial_{\mu} - P^{l}_{h\mu}(\mathbf{x}, \boldsymbol{\nu})\nu^{h}\,\partial'_{l})\Psi(\mathbf{x}, \boldsymbol{\nu}) = mc\Psi(\mathbf{x}, \boldsymbol{\nu}) \tag{3.8}$$

where it is to be remembered that the first and second operators on the lhs operate on the first and second indices of the bispinor $\Psi(\mathbf{x}, \boldsymbol{\nu})$, respectively.

4. DIRAC EQUATION IN ROBERTSON–WALKER SPACE-TIME

It is possible to separate the wave function in the Finsler space that we are considering, i.e., for which

$$g(\mathbf{x}) \equiv F(t) = \exp(\pm b_0x^0), \quad (b_0x^0)^n, \quad \text{or} \quad (1 + b_0x^0)^n, \qquad x^0 = ct$$

It is found that for such cases the connection coefficients $P^{l}_{h\mu}(\mathbf{x}, \boldsymbol{\nu})$ are separated as $P^{l}_{h\mu}(\mathbf{x}, \boldsymbol{\nu}) = \zeta(t)P^{l}_{h\mu}(\boldsymbol{\nu})$, where $2b_0c\zeta(t) = F'(t)/F(t)$. Also, the Dirac matrices $\gamma^{\mu}(\mathbf{x})$ for the curved space-time manifold are connected with the flat-space Dirac matrices γ^{a} through the vierbein fields V^{μ}_{a} by the relations

$$\gamma^{\mu}(\mathbf{x}) = V_a^{\mu}(\mathbf{x})\gamma^a \qquad \text{and} \qquad \gamma_{\mu}(\mathbf{x}) = V_{\mu}^a(\mathbf{x})\gamma_a \tag{4.1}$$

where $V_{\mu}^a(\mathbf{x})$ are the inverse vierbein fields, such that

$$V_a^{\mu}(\mathbf{x})V_{\mu}^b(\mathbf{x}) = \delta_a^b \tag{4.2}$$

In the present case the vierbein fields are diagonal and the diagonal elements are given by

$$V_{\mu}^{\mu}(\mathbf{x}) = (F(t))^{-1/2} = e(t) \quad \text{(say)} \tag{4.3}$$

Equation (3.8) then becomes

$$i\hbar e(t)\gamma^{\mu}(\partial_{\mu} - \zeta(t)P_{h\mu}^{l}(\boldsymbol{\nu})\nu^h\, \partial_l')\Psi(\mathbf{x}, \boldsymbol{\nu}) = mc\Psi(\mathbf{x}, \boldsymbol{\nu}) \tag{4.4}$$

Either the separable wave function $\Psi(\mathbf{x}, \boldsymbol{\nu})$ may be regarded as a matrix product of $\Psi(\mathbf{x})$ and $\varnothing^T(\boldsymbol{\nu})$, i.e.,

$$\Psi(\mathbf{x}, \boldsymbol{\nu}) = \Psi(\mathbf{x})\varnothing^T(\boldsymbol{\nu}) = \begin{pmatrix} \Psi_1(\mathbf{x}) \\ \Psi_2(\mathbf{x}) \\ \Psi_3(\mathbf{x}) \\ \Psi_4(\mathbf{x}) \end{pmatrix} \left(\varnothing_1(\boldsymbol{\nu}), \varnothing_2(\boldsymbol{\nu}), \varnothing_3(\boldsymbol{\nu}), \varnothing_4(\boldsymbol{\nu})\right) \tag{4.5}$$

[i.e., $\Psi_{ij}(\mathbf{x}, \boldsymbol{\nu}) = \Psi_i(\mathbf{x})\varnothing_j(\boldsymbol{\nu})$]

or the bispinor $\Psi(\mathbf{x}, \boldsymbol{\nu})$ can be viewed as the direct product of two spinors $\Psi(\mathbf{x})$ and $\varnothing(\boldsymbol{\nu})$, i.e.,

$$\Psi(\mathbf{x}, \boldsymbol{\nu}) = \Psi(\mathbf{x}) \otimes \varnothing(\boldsymbol{\nu}) \tag{4.6}$$

In De (1991) we derived the Dirac equation in RW space-time (and also the usual Dirac equation of the Minkowski flat space) for the wave function $\Psi(\mathbf{x})$, and thus $\Psi(\mathbf{x})$ represents the usual spinor for the macrospaces. Also, a class of solutions of the equation for $\varnothing(\boldsymbol{\nu})$ which are homogeneous of degree zero in the directional variables has been obtained together with a mass relation as a byproduct. This relation connects the mass of the particle with a parameter in the metric of the underlying manifold. We point out that if $\psi(\mathbf{x}, \boldsymbol{\nu})$ can be written in the separated form

$$\Psi(\mathbf{x}, \boldsymbol{\nu}) = \Psi_1(\mathbf{x}) \otimes \varnothing(\boldsymbol{\nu}) + \Psi_2(\mathbf{x}) \otimes \varnothing^c(\boldsymbol{\nu}) \tag{4.7}$$

where $\Psi_1(\mathbf{x})$ and $\Psi_2(\mathbf{x})$ are eigenstates of γ^0 with eigenvalues $+1$ and -1, respectively, and $\varnothing(\boldsymbol{\nu})$ and $\varnothing^c(\boldsymbol{\nu})$ satisfy, respectively, the equations

$$\left.\begin{aligned} i\hbar\gamma^{\mu}P_{h\mu}^{l}(\boldsymbol{\nu})\nu^h\, \partial_l'\varnothing(\boldsymbol{\nu}) &= \left(Mc - \frac{3i\hbar b_0}{2}\right)\varnothing(\boldsymbol{\nu}) \\ i\hbar\gamma^{\mu}P_{h\mu}^{l}(\boldsymbol{\nu})\nu^h\, \partial_l'\varnothing^c(\boldsymbol{\nu}) &= \left(Mc + \frac{3i\hbar b_0}{2}\right)\varnothing^c(\boldsymbol{\nu}) \end{aligned}\right\} \tag{4.8}$$

then $\Psi(\mathbf{x}, \boldsymbol{\nu})$ satisfies the Dirac equation in the space-time conformal to the Minkowski flat space. It is given by

$$i\hbar\gamma^{\mu}\, \partial_{\mu}\Psi(\mathbf{x}, \boldsymbol{\nu}) + \frac{3i\hbar b_0}{2}\, \zeta(t)\gamma^0\Psi(\mathbf{x}, \boldsymbol{\nu})$$

$$= \frac{c}{e(t)}\, [m + M\zeta(t)e(t)]\Psi(\mathbf{x}, \boldsymbol{\nu}) \tag{4.9}$$

Consequently, the Dirac equation for the RW space-time can be obtained by a pure-time transformation. Here, the additional mass term M appears as the constant in the process of separation of equation (4.4) and this can be considered as a manifestation of the anisotropy of the microdomain.

By using the vierbeins $V^{\alpha}_{\mu}(X)$ which connect this curved space-time with the local inertial frame (the Minkowski flat space), we can recover the usual Dirac equation

$$i\hbar\gamma^{\alpha}\, \partial_{\alpha}\Psi(\mathbf{x}, \boldsymbol{\nu}) = mc\Psi(\mathbf{x}, \boldsymbol{\nu}) \tag{4.10}$$

when one neglects the extremely small second terms in both the lhs and rhs of (4.9). Of course, one can retain the mass term $M\zeta(t)e(t)$, which has significance in the early universe. In fact, in De (1993a) we considered the matter and entropy productions in the very early universe regarded as a thermodynamically open system by incorporating this mass term. Also, quantum creation of matter was considered in De (1993b). The calculated values of the created matter and entropy and also the photon-to-baryon number ratio in the present universe have been found to be in good agreement with the observational data. For particles which are not constituents in the hadron configuration the "$\boldsymbol{\nu}$-part" wave functions $\varnothing(\boldsymbol{\nu})$ and $\varnothing^c(\boldsymbol{\nu})$ have no other physically relevant manifestations because for such particles $M \neq 0$ in the field (wave) equation and consequently no additional quantum number appears; as we shall see later, the opposite holds for the case with $M = 0$. Actually, in the laboratory space-time (Minkowskian), $\varnothing(\boldsymbol{\nu}) = \varnothing^c(\boldsymbol{\nu}) = \text{const}$ ($b_0 \to 0$). Also, we may regard the "averaged" wave function with a pdf $\chi(\boldsymbol{\nu})$,

$$\Psi(\mathbf{x}) = \int \Psi(\mathbf{x}, \boldsymbol{\nu})\chi(\boldsymbol{\nu})\, d^4\nu \tag{4.11}$$

as the wave function in Minkowski space-time. Clearly, the wave function $\Psi(\mathbf{x})$ satisfies the usual Dirac equation. Of course, one can directly find the Dirac equation for the Minkowski space-time from equation (4.4) with the following separated wave function:

$$\Psi(\mathbf{x}, \boldsymbol{\nu}) = \Psi(\mathbf{x}) \otimes \varnothing(\boldsymbol{\nu}) \tag{4.12}$$

where $\varnothing(\boldsymbol{\nu})$ satisfies the equation

$$i\hbar\gamma^{\mu}P^{l}_{h\mu}(\boldsymbol{\nu})\nu^{h}\,\partial_{l}'\varnothing(\boldsymbol{\nu}) = Mc\varnothing(\boldsymbol{\nu}) \tag{4.13}$$

In the hadron configuration it is assumed that $M = 0$ for the constituents and this gives rise to an "additional" quantum number for the particles, which can generate an internal symmetry of the hadrons. This is considered in the following sections.

5. ADDITIONAL QUANTUM NUMBER IN HADRON CONFIGURATION

As pointed out above, we consider the case $M = 0$ for the constituents in the hadron configuration. First, we seek a class of solutions for $\varnothing(\boldsymbol{\nu})$ and $\varnothing^{c}(\boldsymbol{\nu})$ which are homogeneous of degree zero. In fact, the metric tensors of the Finsler space and the fundamental function are homogeneous functions of degree zero and one, respectively, in the directional arguments. Therefore, one can argue that only this class of homogeneous solutions is physically relevant. For the Finsler space that we are considering, the equations for $\varnothing(\boldsymbol{\nu})$ and $\varnothing^{c}(\boldsymbol{\nu})$ for such a type of solutions become

$$\left.\begin{aligned} i\hbar b_0 \sum_{l=1}^{3} \gamma^{l}\left(\nu^{l}\frac{\partial}{\partial\nu^{0}} + \nu^{0}\frac{\partial}{\partial\nu^{l}}\right)\varnothing(\boldsymbol{\nu}) &= -Mc\varnothing(\boldsymbol{\nu}) + \frac{3i\hbar b_0}{2}\varnothing(\boldsymbol{\nu}) \\ i\hbar b^{0} \sum_{i=1}^{3} \gamma^{l}\left(\nu^{l}\frac{\partial}{\partial\nu^{0}} + \nu^{0}\frac{\partial}{\partial\nu^{l}}\right)\varnothing^{c}(\boldsymbol{\nu}) &= -Mc\varnothing^{c}(\boldsymbol{\nu}) - \frac{3i\hbar b_0}{2}\varnothing^{c}(\boldsymbol{\nu}) \end{aligned}\right\} \tag{5.1}$$

In De (1991) we found the solutions, whose general form is as follows:

$$\varnothing(\boldsymbol{\nu}) = \left(\frac{\sqrt{3}\nu^{0} - i\sum_{k=1}^{3}\gamma^{k}\nu^{k}}{\sqrt{3}\nu^{0} + i\sum_{k=1}^{3}\gamma^{k}\nu^{k}}\right)^{D}\omega^{b} \tag{5.2}$$

where D is a complex number and ω^{b} is a four-component spinor independent of $\boldsymbol{\nu}$. A relation between the mass M and the parameter b_0 has also been found. It is given by

$$2\sqrt{3}\hbar b_0 D = Mc \mp 3i\hbar b_0/2 \tag{5.3}$$

[the negative and positive signs on the rhs are for $\varnothing(\boldsymbol{\nu})$ and $\varnothing^{c}(\boldsymbol{\nu})$, respectively].

There is an option in choosing the real part of D. In our previous article, it was chosen as $\mathrm{Re}\{D\} = 0$ or 1 (it is neither very large nor very small and thus this choice is a natural one). As mentioned above, $\mathrm{Re}\{D\} = 0$ for particles within a hadron configuration and for free particles $\mathrm{Re}\{D\} = 1$. The imaginary part of D is given by $\mathrm{Im}\{D\} = \mp\sqrt{3}/4$ for $\varnothing(\boldsymbol{\nu})$ and $\varnothing^c(\boldsymbol{\nu})$, respectively.

Thus, we get the relation

$$2\sqrt{3}\,\hbar b_0 = Mc \qquad \text{for} \quad \mathrm{Re}\{D\} = 1 \tag{5.4}$$

For the universe we live in, it was shown that this relation gives rise to a connection between the mass $\hat{m}$ of the elementary particle and the Hubble function (De, 1991, 1993a) in the following form:

$$\hat{m} = m[1 + 2\alpha H(T)] \tag{5.5}$$

where m is the "inherent" mass of the particle and $\alpha = 10^{-23}$ sec. Here $H(T)$ is the Hubble function expressed as a function of the cosmological time T. Thus, the mass of the particle is the sum of the constant inherent mass and an epoch-dependent mass which is very small at the present epoch (10^{-41} times the inherent mass). But this second mass term was dominant in the very early history of the universe, that is, in the epoch of times less than 10^{-23} sec in the big bang cosmology.

Now, equations (5.1) for the "$\boldsymbol{\nu}$-part" wave functions $\varnothing(\boldsymbol{\nu})$ and $\varnothing^c(\boldsymbol{\nu})$ of the particles in the hadron configuration become

$$\left.\begin{aligned} i\vec{\gamma}\cdot\vec{D}(\nu^0, \vec{\nu})\varnothing(\boldsymbol{\nu}) &= \hbar\varnothing(\boldsymbol{\nu}) \\ i\vec{\gamma}\cdot\vec{D}(\nu^0, \vec{\nu})\varnothing^c(\boldsymbol{\nu}) &= -\hbar\varnothing^c(\boldsymbol{\nu}) \end{aligned}\right\} \tag{5.6}$$

where

$$\vec{\gamma}\cdot\vec{D} = \sum_{l=1}^{3} \gamma^l D^l \qquad \text{with} \quad D^l = -\frac{2}{3}\, i\hbar\left(\nu^0 \frac{\partial}{\partial \nu^l} + \nu^l \frac{\partial}{\partial \nu^0}\right) \tag{5.7}$$

The components of the 3-vector $\vec{D} = (D^1, D^2, D^3)$ satisfy the same commutation relations that are satisfied by the "boost" generators K_j (in the usual coordinate representation) (Perl, 1974). In a sense this vector $\vec{D}$ can be thought of as the boost generator in the tangent space of the Finsler space.

Now, the wave function $\Psi(\mathbf{x}, \boldsymbol{\nu})$, as separated above, has two parts, $\Psi_1(\mathbf{x}) \otimes \varnothing(\boldsymbol{\nu})$ and $\Psi_2(\mathbf{x}) \otimes \varnothing^c(\boldsymbol{\nu})$. In the "rest" frame of the particle these two parts are related to each other as particle and antiparticle. In fact, the antiparticle state $\Psi^c(\mathbf{x}, \boldsymbol{\nu})$ corresponding to a particle state with $\Psi(\mathbf{x}, \boldsymbol{\nu}) = \Psi(\mathbf{x}) \otimes \varnothing(\boldsymbol{\nu})$ is given by

$$\Psi^c(\mathbf{x}, \boldsymbol{\nu}) = i\gamma^2\Psi^*(\mathbf{x}) \otimes i\gamma^2\varnothing^*(\boldsymbol{\nu}) \equiv \Psi^c(\mathbf{x}) \otimes \varnothing^c(\boldsymbol{\nu}) \tag{5.8}$$

It is evident that $\emptyset(\boldsymbol{\nu})$ and $\emptyset^c(\boldsymbol{\nu})$ satisfy equations (5.6). Also, the equation satisfied by $\emptyset^c(\boldsymbol{\nu}) \equiv \emptyset^c(\nu^0, \vec{\nu})$ is also satisfied by $\emptyset(\nu^0, -\vec{\nu})$, since $\vec{D}(\nu^0, -\vec{\nu}) = -\vec{D}(\nu^0, +\vec{\nu})$. Thus,

$$\emptyset^c(\nu^0, \vec{\nu}) \propto \emptyset(\nu^0, -\vec{\nu}) \tag{5.9}$$

It should be noted that equations (5.6) for $\emptyset$ and $\emptyset^c$ are independent of the parameter b_0 (as also of M, since $M = 0$) and when one makes $b_0 \to 0$ in the transition from the comoving coordinates of RW space-time to the local inertial frame (the Minkowski flat space-time) these equations are unaffected and continue to hold. Now, there are two linearly independent solutions for the "$\boldsymbol{\nu}$-part" wave function $\emptyset(\vec{\nu})$ (suppressing the dependence on ν^0, which will be irrelevant in the subsequent discussions) that satisfy the first equation of (5.6), which is, in the units $c = \hbar = 1$,

$$i\vec{\gamma}\cdot\vec{D}\emptyset = \emptyset \tag{5.10}$$

Let us denote them by $\emptyset(\vec{\nu})$ and $\overline{\emptyset}(\vec{\nu})$. Again,

$$i\vec{\gamma}\cdot\vec{D} = -(\gamma^1\gamma^2\gamma^3)(\vec{\Sigma}\cdot\vec{D}) = -(\vec{\Sigma}\cdot\vec{D})(\gamma^1\gamma^2\gamma^3) \tag{5.11}$$

As the three operators $i\vec{\gamma}\cdot\vec{D}$, $\gamma^1\gamma^2\gamma^3$, and $\vec{\Sigma}\cdot\vec{D}$ are mutually commuting, one can construct simultaneous eigenstates of them. Thus, we can take $\emptyset$ and $\overline{\emptyset}$ to be the eigenstates of $\gamma^1\gamma^2\gamma^3$ and $\vec{\Sigma}\cdot\vec{D}$. Let $\emptyset$ and $\overline{\emptyset}$ be the eigenstates of $\gamma^1\gamma^2\gamma^3$ with eigenvalues $+1$ and -1, respectively. (It is easily seen that eigenvalues of $\gamma^1\gamma^2\gamma^3$ and $\vec{\Sigma}\cdot\vec{D}$ are only $+1$ and -1.) As $\emptyset$ and $\overline{\emptyset}$ are linearly independent eigenstates, one can always make them to have eigenvalues $+1$ and -1, respectively. Therefore, we have

$$\vec{\Sigma}\cdot\vec{D}\emptyset(\vec{\nu}) = -\emptyset(\vec{\nu}), \qquad \vec{\Sigma}\cdot\vec{D}\overline{\emptyset}(\vec{\nu}) = \overline{\emptyset}(\vec{\nu}) \tag{5.12}$$

For the antiparticle $\boldsymbol{\nu}$-part wave function that satisfies the second of equations (5.6) one can have two linearly independent solutions $\emptyset^c(\vec{\nu})$ and $\overline{\emptyset}^c(\vec{\nu})$, where $\emptyset^c = i\gamma^2\emptyset^*$ and $\overline{\emptyset}^c = i\gamma^2\overline{\emptyset}^*$. Obviously,

$$\left.\begin{aligned} \gamma^1\gamma^2\gamma^3\emptyset^c(\vec{\nu}) &= -\emptyset^c(\vec{\nu}), \qquad & \gamma^1\gamma^2\gamma^3\overline{\emptyset}^c(\vec{\nu}) &= \overline{\emptyset}^c(\vec{\nu}) \\ \vec{\Sigma}\cdot\vec{D}\emptyset^c(\vec{\nu}) &= -\emptyset^c(\vec{\nu}), \qquad & \vec{\Sigma}\cdot\vec{D}\overline{\emptyset}^c(\vec{\nu}) &= \overline{\emptyset}^c(\vec{\nu}) \end{aligned}\right\} \tag{5.13}$$

As we have seen from the discussion leading to equation (5.9), $\emptyset(\vec{\nu})$ and $\emptyset(-\vec{\nu})$ satisfy two equations (5.6) for $\emptyset(\vec{\nu})$ and $\emptyset^c(\vec{\nu})$, respectively, and the transformation $\vec{\nu} \to -\vec{\nu}$ does not change the eigenvalues for the eigenstates of $\gamma^1\gamma^2\gamma^3$, so we can always take $\emptyset(-\vec{\nu}) = \overline{\emptyset}^c(\vec{\nu})$ and $\overline{\emptyset}(-\vec{\nu}) = \emptyset^c(\vec{\nu})$ (with the constants of proportionality chosen to be unity without loss of generality). From the solutions of $\emptyset$ and $\emptyset^c$ as given explicitly in (5.2), this choice can also be verified. Thus, we have

$$\left.\begin{aligned}\varnothing^c(\vec{\nu}) &= i\gamma^2\varnothing^*(\vec{\nu}) = \overline{\varnothing}(-\vec{\nu})\\ \overline{\varnothing}^c(\vec{\nu}) &= i\gamma^2\overline{\varnothing}^*(\vec{\nu}) = \varnothing(-\vec{\nu})\end{aligned}\right\} \tag{5.14}$$

Table I summarizes these results. Here it is emphasized that $\varnothing(\vec{\nu})$ and $\varnothing(-\vec{\nu})$ represent particle and corresponding antiparticle states (as the $\boldsymbol{\nu}$ parts of the fields) and also these are both eigenstates of $\gamma^1\gamma^2\gamma^3$ with the same eigenvalue. Likewise, $\overline{\varnothing}(\vec{\nu})$ and $\overline{\varnothing}(-\vec{\nu})$ are related as the particle and corresponding antiparticle states having the same eigenvalue of $\gamma^1\gamma^2\gamma^3$ since those are also eigenstates of it. Such a choice is necessary because $\vec{\nu} \to -\vec{\nu}$ makes a particle state into an antiparticle state and vice versa but leaves $\gamma^1\gamma^2\gamma^3\varnothing = \varnothing$ or $\gamma^1\gamma^2\gamma^3\varnothing = -\varnothing$ unchanged. Thus, either $\gamma^1\gamma^2\gamma^3\varnothing = \varnothing$ or $\gamma^1\gamma^2\gamma^3\varnothing = -\varnothing$ can be regarded as the "constraint" on the field (wave) function. A similar situation arises for the Weyl two-component theory of neutrinos. The space reflection $\vec{x} \to -\vec{x}$ makes a particle or antiparticle state into a "nonphysical state." This is because parity is violated in this theory based on the Minkowski space-time, which is isotropic in nature. On the contrary, the space-time of the present consideration is anisotropic, which is manifested in the transition from particle to antiparticle states or vice versa by the "reflection" $\vec{\nu} \to -\vec{\nu}$ in the tangent space of the anisotropic Finsler space.

Now, it is evident from Table I that the particle and antiparticle states with the $\boldsymbol{\nu}$-part wave functions $\varnothing(\vec{\nu})$ and $\varnothing(-\vec{\nu})$ [and also $\overline{\varnothing}(\vec{\nu})$ and $\overline{\varnothing}(-\vec{\nu})$] are eigenstates of an "internal helicity" $\vec{S}\cdot\vec{D}$, where $\vec{S} = \frac{1}{2}\vec{\Sigma}$, the internal spin angular momentum, with opposite eigenvalues $\pm 1/2$. Thus, in the hadron configuration the particle and antiparticle are in internal (spin) angular momentum states having opposite helicities. This relation between the fermion number and internal helicity gives rise to a preferred direction (as we cannot make simultaneous eigenstates for the three operators, the three mutually noncommuting components of $\vec{S} = \frac{1}{2}\vec{\Sigma}$, one has to choose one direction, say the 3-axis, as the preferred direction). Because of this preferred direction in the tangent space of the Finsler space of the microdomain, this space-time is manifestly anisotropic, that is, anisotropic from the physical point of view.

Table I

	"$\boldsymbol{\nu}$ part" of the wave function	Eigenvalues of $\vec{\Sigma}\cdot\vec{D}$	Eigenvalues of $\gamma^1\gamma^2\gamma^3$
Particle	$\varnothing(\vec{\nu})$	-1	$+1$
	$\overline{\varnothing}(\vec{\nu})$	$+1$	-1
Antiparticle	$\overline{\varnothing}(-\vec{\nu}) = \varnothing^c(\vec{\nu})$	-1	-1
	$\varnothing(-\vec{\nu}) = \overline{\varnothing}^c(\vec{\nu})$	$+1$	$+1$

This preferred direction yields a relation between fermion number and the helicity (internal) states and consequently an additional conserved quantum number arises. This internal quantum number can generate the internal symmetry of hadrons. This is considered in the following section.

6. INTERNAL SYMMETRY OF HADRONS

Let us use the following representation of the γ-matrices:

$$\left.\begin{aligned} &\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \overline{\sigma}_\mu & 0 \end{pmatrix}, \qquad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &\overline{\sigma}^\mu = \sigma_\mu = (1, -\vec{\sigma}), \qquad \sigma^\mu = \overline{\sigma}_\mu = (1, +\vec{\sigma}), \qquad \mu = 0, 1, 2, 3 \end{aligned}\right\} \tag{6.1}$$

where $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are Pauli spin matrices. Now, the wave function in the Finsler space, $\Psi(\mathbf{x}, \boldsymbol{\nu})$, when separated as in (4.7), satisfies the Dirac equation in Minkowski flat space-time (also in RW space-time). There, the functions $\Psi_1(\mathbf{x})$ and $\Psi_2(\mathbf{x})$ are eigenstates of γ^0 with eigenvalues $+1$ and -1, respectively. Also, we can have two linearly independent eigenstates of γ^0 having the same eigenvalue and therefore we have the following two sets of eigenstates of γ^0 in the above representation:

$$\left.\begin{aligned} \Psi_1 &= \begin{pmatrix} \chi_1 \\ \chi_1 \end{pmatrix}, \qquad \Psi_2 = \begin{pmatrix} \chi_2 \\ -\chi_2 \end{pmatrix} \\ \overline{\Psi}_1 &= \begin{pmatrix} \overline{\chi}_1 \\ \overline{\chi}_1 \end{pmatrix}, \qquad \overline{\Psi}_2 = \begin{pmatrix} \overline{\chi}_2 \\ -\overline{\chi}_2 \end{pmatrix} \end{aligned}\right\} \tag{6.2}$$

where χ_1, χ_2, $\overline{\chi}_1$, and $\overline{\chi}_2$ are two-component semispinors. From these, one can form the Dirac spinors $\Psi_1(\mathbf{x}, \boldsymbol{\nu})$ and $\Psi_2(\mathbf{x}, \boldsymbol{\nu})$ which may be related to each other as particle and antiparticle as follows:

$$\left.\begin{aligned} \Psi_1(\mathbf{x}, \boldsymbol{\nu}) &= \begin{pmatrix} \chi_1 \\ \chi_1 \end{pmatrix} \otimes \varnothing(\vec{\nu}) + \begin{pmatrix} \chi_2 \\ -\chi_2 \end{pmatrix} \otimes \overline{\varnothing}(-\vec{\nu}) \\ &= \begin{pmatrix} \chi_1 \otimes \varnothing(\vec{\nu}) + \chi_2 \otimes \overline{\varnothing}(-\vec{\nu}) \\ \chi_1 \otimes \varnothing(\vec{\nu}) - \chi_2 \otimes \overline{\varnothing}(-\vec{\nu}) \end{pmatrix} \\ \Psi_2(\mathbf{x}, \boldsymbol{\nu}) &= \begin{pmatrix} \overline{\chi}_1 \\ \overline{\chi}_1 \end{pmatrix} \otimes \overline{\varnothing}(\vec{\nu}) + \begin{pmatrix} \overline{\chi}_2 \\ -\overline{\chi}_2 \end{pmatrix} \otimes \varnothing(-\vec{\nu}) \\ &= \begin{pmatrix} \overline{\chi}_1 \otimes \overline{\varnothing}(\vec{\nu}) + \overline{\chi}_2 \otimes \varnothing(-\vec{\nu}) \\ \overline{\chi}_1 \otimes \overline{\varnothing}(\vec{\nu}) - \overline{\chi}_2 \otimes \varnothing(-\vec{\nu}) \end{pmatrix} \end{aligned}\right\} \tag{6.3}$$

It is easily seen (cf. Table I) that $\Psi_1(\mathbf{x}, \boldsymbol{\nu})$ and $\Psi_2(\mathbf{x}, \boldsymbol{\nu})$ are the eigenstates of the internal helicity operator with eigenvalues $-1/2$ and $+1/2$, respectively.

Budini (1979) suggested that one can generate isospin algebra from the conformal reflection group. The simplest conformally covariant spinor field equation postulated as an $O(4, 2)$ covariant equation in a pseudo-Euclidean manifold $M^{4,2}$ is of the form

$$\left(\Gamma_a \frac{\partial}{\partial \eta_a} + m\right)\xi(\eta) = 0 \tag{6.4}$$

where m is a constant matrix and $\xi(\eta)$ is an eight-component spinor field. Here, the elements of the Clifford algebra Γ_a are the basis unit vectors of $M^{4,2}$. In the fundamental representation where the Γ_a are represented by the 8×8 matrices of the form

$$\Gamma_a = \begin{pmatrix} 0 & \Xi \\ H & 0 \end{pmatrix} \tag{6.5}$$

the conformal spinors ξ can be expressed as

$$\xi = \begin{pmatrix} \varnothing_1 \\ \varnothing_2 \end{pmatrix} \tag{6.6}$$

in which $\varnothing_1$ and $\varnothing_2$ are Cartan semispinors (Cartan, 1966). In this basis, equation (6.4) becomes equivalent in the Minkowski space $M^{3,1}$ to the coupled equations

$$\left.\begin{aligned} i\gamma^\mu \partial_\mu \varnothing_1 &= -m\varnothing_2 \\ i\gamma^\mu \partial_\mu \varnothing_2 &= -m\varnothing_1 \end{aligned}\right\} \tag{6.7}$$

Also, it is possible to obtain from (6.4) a pair of standard Dirac equations in the Minkowski space by a unitary transformation given by

$$C\xi = \xi^{\mathrm{D}} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \qquad C^{-1}\Gamma_\mu C = \Gamma_\mu^{\mathrm{D}} = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & \gamma_\mu \end{pmatrix} \tag{6.8}$$

where

$$C = \begin{pmatrix} L & R \\ R & L \end{pmatrix}, \qquad \text{with} \quad L = \frac{1}{2}(1 + \gamma_5), \quad R = \frac{1}{2}(1 - \gamma_5) \tag{6.9}$$

in the representation (6.1) of γ_5. The superscript D stands for the "Dirac basis." In this Dirac basis Ψ_1, Ψ_2 each satisfies the usual Dirac equation.

An important point should be noted here: the space or time reflection interchanges $\varnothing_1$ and $\varnothing_2$, but transforms Ψ_1 and Ψ_2 into themselves. Moreover, conformal reflection (inverse radius transformation) interchanges both $\varnothing_1 \leftrightarrow \varnothing_2$ and $\Psi_1 \leftrightarrow \Psi_2$. Also, Ψ_1 and Ψ_2 may represent physical free massive fermions, but $\varnothing_1$ and $\varnothing_2$ do not unless they are massless, as they satisfy the coupled equations.

Now, Budini suggested that one can call a reflection algebra correspondingly to a reflection group an internal symmetry algebra for a given field theory:

(a) If the corresponding reflection group when accompanied by the corresponding coordinate reflections is a covariance group for the equation of motion in the Minkowski space.

(b) If it commutes with the Poincaré Lie algebra and with the space-time reflection algebra.

(c) If the transformation induced by the reflection algebra on the fields leaves the action of the theory invariant.

If the reflection algebra commutes only with the Poincaré algebra but does not commute with the space-time reflection algebra L_4, the algebra may be termed a "restricted" internal symmetry algebra.

Budini (1979) showed an important result in the study of the geometry of hadrons, that the internal symmetry algebra can be generated from the conformal reflection group which contains as a subgroup the Lorentz reflection group L_4 of four elements,

$$L_4 = E, S, T, ST = J \tag{6.10}$$

since $O(3, 1)$ is a subgroup of $O(4, 2)$. Here, E = identity, S = space reflection, T = time reflection, and $ST = J$ = strong reflection. In $M^{4,2}$ space, coordinates are taken as η_1, η_2, η_3, η_5, η_0, η_6 with the metric $(+ + + + - -)$. Here, the reflections

$$\left.\begin{aligned} S_5&: \quad \eta_5 \rightarrow \eta_5' = -\eta_5 \\ T_6&: \quad \eta_6 \rightarrow \eta_6' = -\eta_6 \end{aligned}\right\} \tag{6.11}$$

correspond in the Minkowski space, the inverse radius transformation, and the same $\otimes J$. The Abelian group

$$Cp_6 = E, S_5, T_6, S_5T_6 \tag{6.12}$$

is called the partial conformal reflection group and the total conformal reflection group is given by the direct product

$$C_6 = Cp_6 \otimes L_4 \tag{6.13}$$

The conformal reflection group is represented in the conformal spinor space by the algebra $U_{4,c}$, which may be called the conformal reflection algebra.

Now, for a conformal spinor in the Dirac basis

$$\xi^{D} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

the Lorentz reflection group L_4 when acting on the Dirac spinor ψ_i is isomorphic to a U_2 algebra whose Hermitian elements are given by the matrices 1, γ_0, $i\gamma_0\gamma_5$, γ_5. The transformations S_5, T_6, and S_5T_6 that act on the Dirac doublet of the conformal spinor ξ^D correspond to

$$\left.\begin{aligned} S_5 &\rightarrow \Gamma_5^D \\ T_6 &\rightarrow \Gamma_6^D \\ S_5T_6 &\rightarrow \Gamma_5^D\Gamma_6^D \end{aligned}\right\} \tag{6.14}$$

Thus, the group Cp_6 can be represented by the Lie algebra $U_{4,c}$ and the corresponding real subalgebra SU_2 may be obtained from the Hermitian elements $i\Gamma_5$, Γ_6, $\Gamma_5\Gamma_6$. Then it follows from (6.13) that the group C_6 is isomorphic to the product

$$U_{2,c} \otimes U_{2,c} = U_{4,c} \tag{6.15}$$

The following propositions were proved by Budini:

1. The reflection algebra $U_{2,c}$ corresponding to the partial conformal reflection group Cp_6 is an internal symmetry algebra for the conformal spinor doublets. For massive (but degenerate) components of the doublet, $U_{2,c}$ is maximal.

2. For massless conformal spinors or for a system of massive conformal spinors interacting at very short distances, the direct product of the partial conformal reflection group and the strong reflection in the Minkowski space generates a restricted internal symmetry algebra of order eight which can be put in the form $U_{2c,L} \oplus U_{2c,R}$. This $U_{2c,L} \oplus U_{2c,R}$ algebra may be reduced to two independent SU_2 algebras represented by the eight four-dimensional matrices $L \times \sigma_\mu$, $R \times \sigma_\nu$ [where L and R are given by (6.9)] acting on the two independent doublets of Weyl fields into which the massless conformal spinor or the system of the interacting massive spinors splits at short distances.

Now, in the present formalism the conformal spinors represented by the doublets of Cartan semispinors might be regarded as the constituents of a hadron such that they are in the internal spin angular momentum state of $S = 1/2$. In fact, from the spinors as constructed in (6.3) one can regard $(\xi_2^1)^D$ to be the conformal spinor in the Dirac basis, as Ψ_1 and Ψ_2 satisfy the usual Dirac equation. Also, it was pointed out above that Ψ_1 and Ψ_2 are eigenstates of an internal helicity operator with opposite eigenvalues. Since a four-component spinor with $S_z = +1/2$ and another with $S_z = -1/2$ represent constituents for the particle and antiparticle configuration, the doublet can be treated as an eight-component conformal spinor in the Dirac basis as in the approach by Budini in the generation of internal symmetry algebra. From this doublet in the Dirac basis one can find the conformal spinor in the semispinor basis through the transformation

$$\xi^S = \begin{pmatrix} \varnothing_1 \\ \varnothing_2 \end{pmatrix}^S = C\xi^D \tag{6.16}$$

where C is given by (6.9), since $C^2 = 1$. These four-component Cartan semispinors $\varnothing_1$ and $\varnothing_2$ satisfy the coupled equations in the Minkowski space, and consequently the conformal spinor $\xi(\eta)$ is a pseudo-Euclidean manifold $M^{4,2}$ satisfies equation (6.4). Thus, in the present formalism one can use the excellent argument made by Budini to achieve that the direct product of the partial conformal reflection group and strong reflection in the Minkowski space generates a restricted internal symmetry algebra which can be put in the form $U_{2,L} \oplus U_{2,R}$. The elements of the algebra may be represented by the eight four-dimensional matrices $L \times \sigma_\mu$ and $R \times \sigma_\nu$ that act on the two independent doublets of the Weyl fields into which the conformal spinor splits, and thus two independent SU_2 algebras are represented by them. Further, the fixed internal S_z values for particle and antiparticle states give rise to another quantum number representing the algebra of U_1. These SU_2 and U_1 algebras indicate isospin and hypercharge, respectively, and consequently one can achieve a Lie group structure $SU_3 \to SU_2 \times U_1$ for the internal symmetry of hadrons. A similar consideration was made by Bandyopadhyay (1989) with the assumption of internal $l = 1/2$ orbital angular momentum by introducing a preferred direction such that l_z values 1/2 and $-1/2$ represent particle and antiparticle states. There, the anisotropy is introduced through a magnetic monopole and in this space the internal helicity is connected with the fermion number with only a special choice of the value of $\mu = 1/2$, where μ denotes the measure of anisotropy. Also, the connection between the internal helicity and the fermion number made by Bandyopadhyay in a complexified space-time is valid only for massless particles because the formula for the helicity operator used there does not hold when mass is "generated" by the imaginary part of the space-time. On the other hand, in the present formalism the preferred direction is a natural one and arises through the field (wave) equation in the Finsler space as the physical manifestation of the property of fields in this space-time below a length scale of a fundamental length l, where hadronic matter is extended. Here, we get an internal helicity operator $\vec{S} \cdot \vec{D} \equiv \frac{1}{2}\vec{\Sigma} \cdot \vec{D}$ for the constituents in the hadron configuration. The constituents are in the internal (spin) angular momentum ($\vec{S} = \frac{1}{2}\vec{\Sigma}$) state, where the two opposite eigenvalues $\pm 1/2$ of S_z (measured along $\vec{D}$) represent, respectively, the constituents for the particle and antiparticle configuration. This angular momentum corresponds to a continuous group (the Lie group structure). Thus, instead of half-orbital angular momentum as postulated by Bandyopadhyay, the spin-half angular momentum plays the same role in a natural way in building up the group structure (SU_2) instead of algebra, as argued excellently by Budini. Finally with the one-parameter group U_1 from the additional conserved quantum number that arises in the present consideration we get a Lie group formalism for the internal symmetry of hadrons.

7. CONCLUDING REMARKS

In the analysis above we have achieved a Lie group formalism for the internal symmetry of hadrons regarded as extended (as composed of constituents) in a micro-space-time which is the space-time manifested below a length scale of the order of a fundamental length l. This micro-space-time is anisotropic and is characterized as a Finsler space as described above. Macrospaces like the Minkowski space-time of common experience and the space-time that describes the large-scale structure of the universe can be obtained or regarded as "averaged" space-times. The spinor field equation has been derived from a property of the field in this microspace and through space-time quantization similar to that envisaged and used by Namsrai (1985). In the process of the separation of the field function, a mass relation has also been derived as a consequence and it has been found that this relation has important cosmological implications. The present consideration also provides a consistent dynamics of strong interaction for subatomic particles (De, 1986a,b).

Bandyopadhyay *et al.* (1989), in the consideration of half-orbital angular momentum for the constituents of hadrons, have shown that a baryonic multiplet corresponding to the internal symmetry group SU_3 representing baryons with spin 1/2 can be taken to arise from a mesonic SU_3 multiplet having spin zero with a spinorial constituent having the symmetry group U_1. Thus, in a sense, U_1 may be regarded as a baryon-number-generating group and consequently one has

$$SU(3)_{\text{baryon}} \subset U(3) = SU(3)_{\text{meson}} \otimes U_1$$

In this way the meson–baryon mass difference is obtained. In the present consideration, the internal helicity for the constituents of hadrons has been obtained and we have seen that this internal spin 1/2 can well replace the postulated half-orbital angular momentum in building up the internal symmetry of hadrons. As pointed out above, this internal helicity is a natural consequence of the present Finsler space approach to hadronic matter extension. Again, from this concept of spin-1/2 internal angular momentum the meson–baryon mass difference can also be derived exactly as by Bandyopadhyay *et al.* (1989), but not with the consideration of an *ad hoc* $l = 1/2$ internal angular momentum for the constituents. The relation between the fermion number and the internal helicity arises from the field (wave) equation of the field (wave) function $\Psi(\mathbf{x}, \boldsymbol{\nu})$ in Finsler space. Also, in the macrodomain the field satisfies the usual Dirac equation. Thus, the field (wave) function represents a usual spinor and the present consideration emerges as a consistent space-time approach for the hadron structure and dynamics.

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